F-Susy And The Three States Potts Model

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Abstract

In view of its several involvements in various physical and mathematical contexts, 2D-fractional supersymmetry (F-susy) is once again considered in this work. We are, for instance, interested to study the three states Potts model ($k = 3$) which represents with the tricritical Ising model ($k = 2$) the two leading examples of more general spin $1/k$ fractional supersymmetric theories.

1 Introduction

Fractional spin symmetries (FSS) [1, 2, 3], which deals with exotic particles, is an important subject that emerges remarkably twenty years ago in coincidence with the growing interest in high energy and condensed matter physics through quantum field theory [4], conformal symmetries [5] and string theory [6].

These symmetries are specific for two-dimensional theories and play a pioneering role in the study of $D = 2$ conformal field theories and integrable $\Phi_{1,3}$ deformation of minimal models [7]. Well known examples are given by the standard $D = 2$ supersymmetry generated by spin $1/2$ charge operators $Q_{\pm 1/2}$ and the superconformal symmetry exhibiting an infinite number of half integer constants of motion [8].

Other non common examples are the $c = 1 - \frac{6}{p(p+1)}$ minimal models containing among their $\frac{p(p-1)}{2}$ primary fields a spin $\frac{(p+2)}{2}$ conformal field which, combined with the energy momentum tensor, generate a kind of generalized superconformal symmetry.

Recall that the usual superconformal invariance is generated by a spin $3/2$ conserved current in addition to the Virasoro current of spin 2 which give rise, in Laurent modes, to the Neveu-Schwarz and Ramond superconformal algebras. On the other hand, following [9], the $\phi_{1,3}$ deformation of the $c = 6/7$ tricritical three states Potts model exhibits a similar behavior as the $\phi_{1,3}$ deformation of the $c = 7/10$ tricritical Ising model [10]. Both of them admit fractional spin constants of motion namely $Q_{\pm 1/3}^{\pm}$ and $Q_{\pm 1/2}$ surviving after the perturbation and satisfying

$$Q_{\pm 1/2}^2 = P_{\pm 1}, \tag{1}$$

and

$$Q_{1/3}^{-3} = Q_{1/3}^{+3} = P_1 \tag{2}$$

$$Q_{-1/3}^{-3} = Q_{-1/3}^{+3} = P_{-1} \tag{3}$$

where $P_\mu = (P_1, P_{-1})$ is the two dimensional energy momentum vector. The $\pm 1/3$ lower indices and $\pm$ upper ones carried by the $Q_{\pm 1/3}$'s are respectively the values of the spin and the charges of the Z_3 automorphism symmetry of eqs(2-3). Note also that the above equations are particular examples of more general fractional spin $s = \pm 1/k$ equations generalizing the $D = 2$ supersymmetry algebra and reading as

$$Q_s^{\pm k} = P_{ks} \tag{4}$$

Setting $k = 2$ and $k = 3$, one gets respectively the standard $D = 2 N = 1$ supersymmetry and the leading generalized one eqs(2-3). Moreover, it is established that under the $\phi_{1,3}$ deformation, a $c(p)$ minimal model flows to the subsequent $c(p-1)$ conformal one [7] in agreement with the Zamolodchikov c-theory [11] according to which the central charge c is a decreasing function in the space of coupling parameters.

We focus in this work to renew our interest in fractional supersymmetry, although several productions have been made previously, since we believe that much more important results can be extracted. This is also important to give new breath to supersymmetry,

conformal invariance and integrable models as one of the best issues in the history of theoretical and mathematical physics.

We present the $D = 2$ three states Potts model as been the first non trivial physical model corresponding to a fractional deformation of the standard supersymmetry and show later how the $D = 2(1/3, 1/3)$ can be it's underlying invariance. We are projecting through this first presentation among a series of fourth coming works to shed new insights towards understanding well these exotic symmetries and their possible incorporation in various modern topics of theoretical and mathematical physics.

2 $D = 2$ Three states Potts Model

It's now well known that the $c = 6/7$ conformal theory and more particularly the three critical Potts model (TPM), admits several several infinite dimensional symmetries. The first kind is given by the conformal symmetry whose generator $T_{\mu\nu}, \mu, \nu = z, \bar{z}$ is nothing but the spin 2 energy momentum tensor which is symmetric and traceless. The second infinite symmetry is generated by the so called conserved W-currents. Combined with the spin 2-conformal current, the W-currents generate a huge infinite symmetry known as the W-symmetry [12].

The famous example of this non standard symmetry is given by the Zamolodchikov algebra generated by T of spin 2 and W of spin 3 conformal currents. The common property of W-symmetries is the fact that they are associated to a non standard Lie algebra structure. W-algebra is a non linear symmetry containing the conformal one as a particular case and is known as the mediator of the integrability of the $\phi_{1,2}$ magnetic deformation of the $c = 6/7$ critical theory [9].

The third class of infinite symmetries of the TPM is generated by fractional spin $4/3$ conserved currents $G_{4/3}^{\pm}$ and $\bar{G}_{4/3}^{\pm}$ in addition to the energy momentum tensor. We shall refer to this symmetry as the $4/3$-superconformal symmetry in analogy with the standard superconformal symmetry generated by spin $3/2$ current and which we denote as spin $3/2$-superconformal symmetry.

This invariance of the $c = 6/7$ critical theory generalizes in some sense the $N = 1$ spin $3/2$ superconformal symmetry of the tricritical Ising model (TIM) having central charge c equal to $7/10$. Recall that TIM and TPM are respectively given by the fourth and sixth levels of the minimal series

$$c(p) = 1 - \frac{6}{p(p+1)}; p = 3, 4, ..., \tag{5}$$

They appear also as the leading conformal theories of the $N = 1$ spin $3/2$ superconformal discrete series

$$c(m) = \frac{3}{2}(1 - \frac{8}{m(m+2)}); m = 3, 4, ..., \tag{6}$$

and the spin $4/3$ superconformal discrete one

$$c(m) = 2(1 - \frac{12}{m(m+4)}); m = 3, 4, ..., \tag{7}$$

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Note also that eqs(5-7) may be regrouped into a two integers discrete series as [13]

$$c(m,n) = \frac{3n}{n+2}(1 - \frac{2(n+2)}{m(m+n)}); n = 1, 2, ..., \tag{8}$$

For $n = 1$, one recover the unitary minimal models see eq(5). Putting $n = 2$ and $n = 4$ in the above relation by keeping the integer m free, we obtain respectively the spin 3/2-superconformal and 4/3-superconformal theories. However, letting the integer n free and taking $m = 3$ we get

$$c(3,n) = 1 - \frac{6}{(n+2)(n+3)}; n = 1, 2, ..., \tag{9}$$

which corresponds to the minimal unitary series eq(5) once we set $n + 2 = p$.
From this overlapping of the above discrete series, we deduce that $N = 0$ conformal models of eq(5) admit extra symmetries since they appear as special critical models of the $c(m,n)$ theories.

Another interesting aspect exhibited by the TIM and TPM and more generally the $c(k) = 1 - \frac{3}{k(2k+1)}; k = 2, 3, ...,$ conformal models is the integrability of their $\phi_{1,3}$ deformation, see the last ref. in [7]. It's shown there that the thermal perturbation of the $c = 6/7$ model induces an off critical spin 1/3 supersymmetric algebra surviving after the $\phi_{1,3}$ perturbation. This is a finite dimensional symmetry generated by conserved charges $Q^{\pm}_{+1/3}$ and $Q^{\pm}_{-1/3}$ carrying fractional spin $s = \pm 1/3$ and non vanishing Z_3 charges, we have

$$Q^{-3}_{1/3} = P_1, Q^{-3}_{-1/3} = P_{-1} \tag{10}$$
$$Q^{+3}_{1/3} = P_1, Q^{+3}_{-1/3} = P_{-1} \tag{11}$$

and

$$Q^{-}_{1/3}Q^{-}_{-1/3} - \bar{q}Q^{-}_{-1/3}Q^{-}_{1/3} = \Delta^{(-,-)} \tag{12}$$
$$Q^{+}_{1/3}Q^{+}_{-1/3} - qQ^{+}_{-1/3}Q^{+}_{1/3} = \Delta^{(+,+)} \tag{13}$$
$$Q^{-}_{1/3}Q^{+}_{-1/3} - \bar{q}Q^{+}_{-1/3}Q^{-}_{1/3} = \Delta^{(-,+)} \tag{14}$$
$$Q^{+}_{1/3}Q^{-}_{-1/3} - qQ^{-}_{-1/3}Q^{+}_{1/3} = \Delta^{(+,-)} \tag{15}$$

where P_3 is the usual 2d energy momentum vector and $\Delta^{(r_1,r_2)}, r_1, r_2 = \pm 1$ are topological charges. the parameter q is such that $q^3 = 1$ chosen as $q = exp(2i\pi/3)$. It can be thought of as the deformation parameter of the $U_q(sl(2))$ quantum enveloping algebra of $sl(2)$ [14]. The parameter q describes also the generator of the Z_3 discrete abelian group. Denoting by $S^\star$ the critical action of TPM and by S its deformation $\phi_{h,\bar{h}} = \phi_{1,3} \otimes \phi_{1,3}; h, \bar{h} = 5/7$ namely

$$S = S^* + \lambda \int d^2z \phi_{5/7,5/7} \tag{16}$$

where λ is the perturbation parameter, it was shown that the algebra eq(10-15) is a symmetry of the above deformed theory. The conserved charges $Q^{\pm}_s$, $\Delta^{(r_1,r_2)}$ and $P_{3s}, s = \pm 1/3$ are realized as follows

$$Q^{\pm}_{1/3} = \int \left[dz G^{\pm}(z,\bar{z}) + d\bar{z}\Gamma^{\pm}(z,\bar{z})\right] \tag{17}$$

$$Q^{\pm}_{-1/3} = \int \left[d\bar{z}\bar{G}^{\pm}(z,\bar{z}) + dz\bar{\Gamma}^{\pm}(z,\bar{z})\right] \tag{18}$$

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and

$$\Delta^{(+,+)} = \int \left[dz\partial + +d\bar{z}\bar{\partial} \right] \varphi^{(+,+)} \tag{19}$$

$$\Delta^{(-,-)} = \int \left[dz\partial + d\bar{z}\bar{\partial} \right] \varphi^{(-,-)} \tag{20}$$

$$\Delta^{(+,-)} = \int \left[dz\partial + d\bar{z}\bar{\partial} \right] \varphi^{(+,-)} \tag{21}$$

$$\Delta^{(-,+)} = \int \left[dz\partial + d\bar{z}\bar{\partial} \right] \varphi^{(-,+)} \tag{22}$$

$$\tag{23}$$

and

$$P_1 = \int \left[dzT + d\bar{z}\Theta \right] \tag{24}$$

$$P_{-1} = \int \left[d\bar{z}\bar{T} + dz\Theta \right] \tag{25}$$

The conformal field $\phi_{h,\bar{h}}^{(r,\bar{r})}(z,\bar{z}) = \phi_h^r(z) \otimes \phi_{\bar{h}}^{(\bar{r})}(\bar{z})$, with $r,\bar{r} = 0,1,2(mod3)$ are the $Z_3 \times \bar{Z}_3$ (left right) charges, appearing in the above equations, which are built as: the $4/3$ supersymmetric currents $G_{4/3}^{\pm}$ (resp. $\bar{G}_{-4/3}^{\pm}$) which carry only a left (resp. right) Z_3 charge read as

$$G = \phi_{4/3,0}^{(\pm,0)}, \bar{G}^{\pm} = \phi_{0,4/3}^{(0,\pm)} \tag{26}$$

The magnetic order parameter fields $\varphi^{(+,+)}$ and its conjugate $\varphi^{(-,-)}$ are given by

$$\varphi^{(+,+)} = \phi_{1/21,1/21}^{(+,+)}, \varphi^{(-,-)} = \phi_{1/21,1/21}^{(-,-)} \tag{27}$$

They carry the same left and right Z_3 charge contrary to the magnetic disorder parameter fields $\varphi^{(+,-)}$ and $\varphi^{(-,+)}$ which read as

$$\phi^{(+,-)} = \phi_{1/21,1/21}^{(+,-)}, \phi^{(-,+)} = \phi_{1/21,1/21}^{(-,+)}. \tag{28}$$

The remaining relevant fields of the TPM involved in eqs(17-25) are

$$\bar{\Gamma}_{-2/3}^{\pm} = \phi_{1/21,1/21}^{(\pm,0)}, \Gamma_{2/3}^{\pm} = \phi_{5/7,1/21}^{(0,\pm)}, D^0 = \phi_{5/7,5/7}^{(0,0)} \tag{29}$$

The field Θ appearing in the two last relations eqs(24-25) is the trace of the conserved energy momentum tensor of the off critical theory. It measures the violation of the scale invariance of the $\phi_{5/7,5/7}$ deformation of the TPM model. It reads then as

$$\Theta \sim \lambda \phi_{5/7,5/7}^{(0,0)} \tag{30}$$

Note that the fields $\phi, \Gamma, \bar{\Gamma}$ and D eqs(27-29) have values of the spin $s = h - \bar{h}$ respectively equal to $0, (1-s), -(1-s), 0$ with $s = 1/3$. Note also that the above field operators share some basic features with the four fields involved in the $N = 1$ spin $1/2$ supersymmetric $\phi_{1,3} = \phi_{3/5,3/5}$ deformation of the TIM [10].
There, these conformal fields have respectively the spin values $0, (1-s), -(1-s)$ and 0

with $s = 1/2$. They belong to the scalar representation of the two dimensional $N = 1$ spin $1/2$ supersymmetric algebra

$$Q_s Q_s + Q_s Q_s = P_{2s}, s = \pm 1/2 \tag{31}$$

$$Q_s Q_{-s} + Q_{-s} Q_s = \Delta \tag{32}$$

This algebra is generated by hermitian charges and admits a field representation analogous to the field realization eqs(24-25) of the off critical spin $1/3$ superalgebra. We have

$$Q_{1/2} = \int dz G + d\bar{z}\Gamma] \tag{33}$$

$$Q_{-1/2} = \int \left[d\bar{z}\bar{G} + dz\bar{\Gamma} \right] \tag{34}$$

$$\Delta = \int \left[dz\partial + + d\bar{z}\bar{\partial} \right] \varphi \tag{35}$$

$$P = \int \left[dzT + d\bar{z}\Theta \right] \tag{36}$$

$$\bar{P} = \int \left[d\bar{z}\bar{T} + dz\Theta \right] \tag{37}$$

We can define the following fields operators $G_{3/2}, \bar{G}_{-3/2}, \phi, F, G_{1/2}$ and $\bar{G}_{-1/2}$ in terms of the field ϕ as follows

$$G_{3/2} = \phi_{3/2,0}, \bar{G}_{-3/2} = \phi_{0,3/2} \tag{38}$$

$$\varphi = \phi_{1/10,1/10}, F = \phi_{3/5,3/5} \tag{39}$$

$$\bar{G}_{-1/2} = \phi_{1/10,3/5}, G_{1/2} = \phi_{3/5,1/10} \tag{40}$$

and where Θ is shown also to be proportional to the perturbation field $F = \phi_{3/5,3/5}$ namely $\Theta = \alpha F$.

3 The Underlying $D = 2(1/3, 1/3)$ Supersymmetry

An important question that emerges when studying exotic fractional symmetries is their superspace representations. In previous works [2], we succeeded to build representations of the $D = 2$ fractional supersymmetric algebra, noted simply as $(1/k, 1/k)$ for $k = 2, 3, \ldots$. The last notation indicates simply the left and right-hand sides of the supersymmetric algebra generated by spin $s = \pm 1/k$ charge operators Q and $\bar{Q}$ satisfying eq(4).

Based on this knowledge and on the fact that the particular choice $k = 2$ reproduce the standard $D = 2$ supersymmetry, we focus in what follows to show how $(1/3, 1/3)$ supersymmetric algebra can be considered as the underlying symmetry of the $D = 2$ three state Potts model.

Recall first that there are few known models that exhibit the $(1/3, 1/3)$ supersymmetric algebra. We quote the $C = 6/7$ minimal models and its $\phi_{1,3}$ deformation. In these cases, the $D = 2(1/3, 1/3)$ superfields are characterized by their spins $s = h - \tilde{h}$ and their scale dimension $\Delta = h + \tilde{h}$. They contain 3×3 component fields depending on the space

coordinates z and $\bar{z}$ and on the extra variable u realizing the topological charge. Setting $h = \tilde{h} = 1/21$, by virtue of the $C = 6/7$ minimal model [7], the scalar superfield $\phi_{1/21,1/21}$ expands in $\theta_{-1/3}$ and $\tilde{\theta}_{1/3}$ series as

$$
\begin{aligned}
\phi_{(1/21,1/21)} &= \varphi_{(1/21,1/21)} + \theta_{-1/3}\psi_{(8/21,1/21)} + \tilde{\theta}_{1/3}\tilde{\psi}_{(1/21,8/21)} \\[2mm]
&+ \theta^2_{-1/3}\chi_{(15/21,1/21)} + \tilde{\theta}^2_{1/3}\tilde{\chi}^2_{(1/21,15/21)} + \theta_{-1/3}\tilde{\theta}_{1/3}\xi_{(8/21,8/21)} \\[2mm]
&+ \theta^2_{-1/3}\tilde{\theta}_{1/3}\lambda_{(15/21,8/21)} + \tilde{\theta}^2_{1/3}\theta_{-1/3}\tilde{\lambda}_{(8/21,15/21)} + \theta^2_{-1/3}\tilde{\theta}^2_{1/3}F_{(15/21,15/21)}.
\end{aligned}
$$
(41)

From this expansion we recognize the fields involved in the TPM discussed in the previous section. Note for instance the last term of this expression namely $F_{(15/21,15/21)}$ is nothing but the $\phi_{1,3}$ field $\phi_{(5/7,5/7)}$ of the TPM or $C = 6/7$ conformal theory. This is a $D = 2(1/3, 1/3)$ supersymmetric invariant quantity exactly as for the field of the $D = 2(1/2, 1/2)$ supersymmetric $C = 7/10$ minimal model.

Based on this presentation, others important aspects of fractional supersymmetries will be presented in our forthcoming works.

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